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Normal structure and moduli of UKK, NUC, and UKK* in Banach spaces

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ABSTRACT

Let X be a Banach space with the closed unit ball $B(X)$. In this paper, by directly extrapolating from the definitions of Uniformly Kadec–Klee (UKK), Nearly Uniformly Convex (NUC) and Weak* Uniformly Kadec–Klee (w*UKK) spaces, we consider the concepts of the modulus of UKK and the modulus of NUC on X, and the modulus of UKK* on the dual space X^* of X. Some new properties of Banach spaces related to reflexivity and normal structure with the values of these moduli are obtained. Among these new results, we prove that if $B(X^*)$ is weak* sequentially compact and $\mathrm{UKK}*((\frac{1}{\mu(X^*)})^-) > 1 - \frac{1}{\mu(X^*)}$ for X^*, then X has weak normal structure, where $\mu(X)$ is the separation measure of $B(X)$.

1. Introduction

Let X be a normed linear space, and let $B(X) = \{x \in X : \|x\| \leq 1\}$, $S(X) = \{x \in X : \|x\| = 1\}$, and $B_\gamma(0) = \{x \in X : \|x\| < \gamma\}$ be the unit ball, the unit sphere, and open ball with radius γ of X respectively. Let X^* be the dual space of X.

Brodskiĭ and Mil'man introduced the concept of normal structure in 1948 [1].

For a reflexive Banach space, the normal structure and weak normal structure coincide.

In 1965, Kirk [2] proved that if a Banach space X has weak normal structure, then it has weak fixed point property, that is, every nonexpansive mapping from a weakly compact and convex subset of X into itself has a fixed point.

For a sequence $\{x_n\} \subseteq B(X)$, let $\mathrm{co}(x_n)$, $\overline{\mathrm{co}}(x_n)$, and $\overline{\mathrm{co}}^w(x_n)$ be the convex hull, the closed convex hull, and the weak closed convex hull of the sequence $\{x_n\}$, respectively. Moreover, let $\mathrm{sep}(x_n) \equiv \inf\{\|x_n - x_m\| : n \neq m\}$. We also write $\xrightarrow{w}$ and $\xrightarrow{w^*}$ for the weak and the weak* convergence respectively, and write $f(a^-)$ for $\lim_{\varepsilon \to a^-} f(\varepsilon)$ for a function $f(\varepsilon)$.

Huff [3] introduced two concepts of Uniformly Kadec–Klee (UKK) spaces and Nearly Uniformly Convex (NUC) spaces on X in 1980.

Definition 1.1 (*[3]*)**.** A Banach space X is called a UKK space (Uniform Kadec–Klee space) if for any $\varepsilon > 0$ there exists $0 < \delta < 1$ such that for any sequence $\{x_n\} \subseteq B(X)$ with $x_n \xrightarrow{w} x$ and $\mathrm{sep}(x_n) \geq \varepsilon$, it follows that $x \in B_\delta(0)$.

Definition 1.2 (*[3]*)**.** A Banach space X is called an NUC space (Nearly Uniform Convex space) if for any $\varepsilon > 0$ there exists $0 < \delta < 1$ such that for any sequence $\{x_n\} \subseteq B(X)$ with $\mathrm{sep}(x_n) \geq \varepsilon$, it follows that $\mathrm{co}(x_n) \bigcap B_\delta(0) \neq \emptyset$.

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He proved the relationship between UKK spaces and NUC spaces, and the relationship between NUC spaces and normal structure:

Theorem 1.3 (*[3]*)**.** *Let X be a Banach space. Then X is an* NUC *space if and only if X is a* UKK *space and reflexive.*

Theorem 1.4 (*[4,3]*)**.** *Every* NUC *space has normal structure.*

Dowling et al. [5] introduced the concept of UKK* spaces in 2008.

Definition 1.5 (*[5]*)**.** Let X be a Banach space. The dual space X^* of X is called a UKK* space (Uniform Kadec–Klee* space) if for any $\varepsilon > 0$ there exists $0 < \delta < 1$ such that for any sequence $\{f_n\} \subseteq B(X^*)$ with $f_n \xrightarrow{w^*} f$ and $\mathrm{sep}(f_n) \geq \varepsilon$, it follows that $f \in B_\delta(0)$ for X^*.

The following result was proved:

Theorem 1.6 (*[5]*)**.** *Let X be a Banach space such that $B(X^*)$ is weak* sequentially compact. If X^* is a* UKK* *space, then X has weak fixed point property.*

Some measures of noncompactness on a Banach space X have been introduced to study the geometrical properties of Banach spaces, see [6–12].

Definition 1.7 (*[6]*)**.** Let A be a bounded set of a metric space. The α-measure, χ-measure, and β-measure of noncompactness of A are defined by

$$\alpha(A) = \inf\{\varepsilon > 0 : A \text{ can be covered by finitely many sets with diameter } \leq \varepsilon\};$$

$$\chi(A) = \inf\{\varepsilon > 0 : A \text{ can be covered by finitely many balls with radii } \leq \varepsilon\};$$

$$\begin{aligned}\beta(A) = \sup\{&\varepsilon > 0 : A \text{ contains an infinite } \varepsilon-\text{separation, that is, there is an infinite subset } C \text{ of } A\\ &\text{such that } d(x,y) \geq \varepsilon \text{ for all } x, y \in C \text{ with } x \neq y\},\end{aligned}$$

respectively.

It is known that if A is a bounded subset of a Banach space, then

$$\chi(A) \leq \beta(A) \leq \alpha(A) \leq 2\chi(A).$$

Definition 1.8. Let X be a Banach space. The α-modulus [6,10], χ-modulus [6,7], and β-modulus [6,9] of noncompact convexity of X are defined by

$$\Delta_\alpha(\varepsilon) = \inf\{1 - \inf_{x\in A}\|x\| : A \text{ is a convex subset of } B(X) \text{ and } \alpha(A) > \varepsilon\};$$

$$\Delta_\chi(\varepsilon) = \inf\{1 - \inf_{x\in A}\|x\| : A \text{ is a convex subset of } B(X) \text{ and } \chi(A) > \varepsilon\};$$

$$\Delta_\beta(\varepsilon) = \inf\{1 - \inf_{x\in A}\|x\| : A \text{ is a convex subset of } B(X) \text{ and } \beta(A) > \varepsilon\},$$

respectively.

The following result was proved:

Theorem 1.9 (*[6, Theorem V.1.7]*)**.** *Let X be a Banach space. If $\Delta_\chi(1) > 0$, then X is reflexive.*

The function $P_X(\varepsilon)$ below was defined by Partington: (it was also considered in [12] but denoted by $\diamond(\varepsilon)$)

Definition 1.10 (*[6,13]*)**.**

$$P_X(\varepsilon) = \inf\{1 - \|x\| : x_n \xrightarrow{w} x, \{x_n\} \subseteq B(X) \text{ and } \mathrm{sep}(x_n) > \varepsilon\}.$$

The following result was proved:

Theorem 1.11 (*[6, Theorem V.1.10, Remark V.1.12]*)**.** *For a reflexive Banach space X, $\Delta_\beta(\cdot) = P_X(\cdot) = \diamond(\cdot)$.*

2. Main results

Lemma 2.1 (*[14]*)**.** *Let X be a Banach space. Suppose that X does not have weak normal structure. Then there exists a weak null sequence $\{y_n\} \subseteq X$ such that for every $k \in \mathbb{N}$,*

$$1 - \frac{1}{k} < \left\| \sum_{j=0}^{l} \lambda_j y_{k+j} - y_{k+l+1} \right\| < 1 + \frac{1}{k}$$

holds for all $l \in \mathbb{N}$ and all $\lambda_1, \ldots, \lambda_l \geq 0$ with $\sum_{j=1}^{l} \lambda_j = 1$.

Then, it is easy to get the following result:

Lemma 2.2. *Let X be a Banach space without weak normal structure. Then for any $0 < \epsilon < 1$, there exists a sequence $\{x_n\} \subseteq S(X)$ with $x_n \xrightarrow{w} 0$, and*

$$1 - \epsilon < \|x_{n+1} - x\| < 1 + \epsilon$$

for sufficiently large n, and any $x \in \mathrm{co}\{x_k\}_{k=1}^n$.

Theorem 2.3 ([15]). *Let X be an infinite dimensional normed space and $\mu(X) = \beta(B(X))$. Then $1 \le \mu(X) \le 2$, and $\mu(l_p) = 2^{\frac{1}{p}}$ where $1 \le p < \infty$.*

By directly extrapolating from the definitions of UKK and NUC spaces, we first consider the concepts of modulus of UKK and modulus of NUC on X as follows:

Definition 2.4. Let X be a Banach space. For $0 < \varepsilon \le \mu(X)$, let

$$\mathrm{UKK}(\varepsilon) = 1 - \delta,$$

where δ is the smallest number in $(0, 1]$ such that $x \in B_\delta(0)$ whenever $\{x_n\} \subseteq B(X)$ satisfies $x_n \xrightarrow{w} x$ and $\mathrm{sep}(x_n) \ge \varepsilon$. For $\mu(X) < \varepsilon \le 2$, let $\mathrm{UKK}(\varepsilon) = 1$. Then the function $\mathrm{UKK}(\varepsilon)$ is called the UKK modulus of X.

Remark 2.5. It follows directly from the definitions that

$$\mathrm{UKK}(\varepsilon) = P_X(\varepsilon) = \diamond(\varepsilon)$$

for $0 \le \varepsilon \le \mu(X)$.

Definition 2.6. Let X be a Banach space. For $0 < \varepsilon \le \mu(X)$, let

$$\mathrm{NUC}(\varepsilon) = 1 - \delta,$$

where δ is the smallest number in $(0, 1]$ such that $\mathrm{co}(x_n) \bigcap B_\delta(0) \neq \emptyset$ whenever $\{x_n\} \subseteq B(X)$ satisfies $\mathrm{sep}(x_n) \ge \varepsilon$. For $\mu(X) < \varepsilon \le 2$, let $\mathrm{NUC}(\varepsilon) = 1$. Then the function $\mathrm{NUC}(\varepsilon)$ is called the NUC modulus of X.

We then consider the following concept for the dual space X^*:

Definition 2.7. Let X^* be a dual of a Banach space X. For $0 < \varepsilon \le \mu(X^*)$, let

$$\mathrm{UKK}^*(\varepsilon) = 1 - \delta,$$

where δ is the smallest number in $(0, 1]$ such that $f \in B_\delta(0)$ whenever $\{f_n\} \subseteq B(X^*)$ satisfies $f_n \xrightarrow{w^*} f$ and $\mathrm{sep}(f_n) \ge \varepsilon$. For $\mu(X^*) < \varepsilon \le 2$, let $\mathrm{UKK}^*(\varepsilon) = 1$. Then the function $\mathrm{UKK}^*(\varepsilon)$ is called the UKK* modulus of X^*.

Example 2.8. (1) $\mathrm{NUC}(2) = 0$ for l_1 space. Let $e_i = (0, \ldots, 0, 1, 0, \ldots)$, where the ith element is 1, be the standard basis of l_1. Then $\mathrm{sep}(e_i) = 2$, and $\mathrm{co}(e_i) \subseteq S(l_1)$. So $\mu(l_1) = 2$ and $\mathrm{NUC}(2) = 0$.
(2) $\mathrm{NUC}(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$ for Hilbert spaces.

We immediately obtain the following easy results.

Proposition 2.9. *Let X be a Banach space.*

(1) *All* UKK *modulus,* NUC *modulus of X, and* UKK* *modulus of X^* are nondecreasing functions.*
(2) *X is a* UKK *space if and only if* $\mathrm{UKK}(\varepsilon) > 0$ *for all $\varepsilon > 0$.*
(3) *X is an* NUC *space if and only if* $\mathrm{NUC}(\varepsilon) > 0$ *for all $\varepsilon > 0$.*
(4) *X^* is a* UKK* *space if and only if* $\mathrm{UKK}^*(\varepsilon) > 0$ *for all $\varepsilon > 0$.*

The following ingenious result was proved by James.

Theorem 2.10 ([16]). *Let X be a Banach space. Then X is not reflexive if and only if for any $0 < \varepsilon < 1$, there are a sequence $\{x_n\} \subseteq S(X)$ and a sequence $\{f_n\} \subseteq S(X^*)$ such that*

(a) *$\langle x_m, f_n\rangle = \varepsilon$ whenever $n \le m$; and*
(b) *$\langle x_m, f_n\rangle = 0$ whenever $n > m$.*

Using the preceding result, we obtain the following result.

Theorem 2.11. *If X is a Banach space with* $\mathrm{NUC}(1^-) > 0$, *then it is reflexive.*

Proof. By the assumption, we can find an $\varepsilon \in (0, 1)$ such that $\mathrm{NUC}(\varepsilon) > 1 - \varepsilon$. Suppose that X is not reflexive. By Theorem 2.10, there are two sequences $\{x_n\} \subseteq S(X)$ and $\{f_n\} \subseteq S(X^*)$ satisfying the two conditions there. Then $\|x_n - x_m\| \geq \langle x_n - x_m, f_n\rangle = \varepsilon$ for $n > m$ and hence $\mathrm{sep}(x_n) = \inf\{\|x_n - x_m\| : n > m\} \geq \varepsilon$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_k \geq 0$ be such that $\sum_{i=1}^k \alpha_i = 1$. Suppose that k_0 is the first number such that $\alpha_{k_0} > 0$. Then $\|\sum_{i=1}^k \alpha_i x_i\| \geq \langle \sum_{i=1}^k \alpha_i x_i, f_{k_0}\rangle = \varepsilon$ and this implies that $\mathrm{co}(x_n) \bigcap B_\varepsilon(0) = \emptyset$. So $\mathrm{NUC}(\varepsilon) < 1 - \varepsilon$, a contradiction. □

The following theorem shows the relation between the moduli $\mathrm{UKK}(\varepsilon)$ and $\mathrm{NUC}(\varepsilon)$:

Theorem 2.12. *Let X be a Banach space. Then*

1. $\mathrm{NUC}(\varepsilon) \leq \mathrm{UKK}(\varepsilon)$ *for any* $0 < \varepsilon < 1$;
2. $\mathrm{NUC}(\varepsilon) = \mathrm{UKK}(\varepsilon)$ *for any* $0 < \varepsilon < 1$ *if X is reflexive;*
3. *For any* $0 < \varepsilon < 1$, $\mathrm{NUC}(\varepsilon) = \alpha > 0$ *if and only if* $\mathrm{UKK}(\varepsilon) = \alpha > 0$ *and X is reflexive.*

Proof. (1) Put $\alpha \equiv \mathrm{NUC}(\varepsilon)$. Let $\eta > 0$ and $\{x_n\} \subseteq S(X)$ be such that $\mathrm{sep}(x_n) \geq \varepsilon$ and $x_n \xrightarrow{w} x$. If $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, then $\mathrm{sep}(x_{n_k}) \geq \varepsilon$ and $x_{n_k} \xrightarrow{w} x$. We can find sequences $\{p_n\}$ and $\{q_n\}$ of positive integers, a sequence $\{\alpha_n\}$ in $[0, 1]$ such that the following conditions are satisfied:

- $1 < p_1 < q_1 < p_2 < q_2 < \cdots < p_j < q_j < \cdots$ for all j;
- $\sum_{i=p_j}^{q_j} \alpha_i = 1$ for all i and j;
- $\sum_{i=p_j}^{q_j} \alpha_i x_i \in B_{1-\alpha+\eta}(0)$ for all j.

It is easy to see that $\sum_{i=p_j}^{q_j} \alpha_i x_i \xrightarrow{w} x$ as $j \to \infty$. Hence $x \in B_{1-\alpha+\eta}(0)$ for any $\eta > 0$ and we have $\mathrm{UKK}(\varepsilon) \geq \alpha$.

(2) Put $\beta \equiv \mathrm{UKK}(\varepsilon)$. Suppose that a sequence $\{x_n\} \subseteq S(X)$ satisfies $\mathrm{sep}(x_n) \geq \varepsilon$. By the reflexivity and passing to an appropriate subsequence but still denoted by $\{x_n\}$, we may assume that $x_n \xrightarrow{w} x \in \overline{\mathrm{co}}^w(x_n) = \overline{\mathrm{co}}(x_n)$. It follows that $x \in \mathrm{co}(x_n) \bigcap B_{1-\beta+\eta}(0)$ for any $\eta > 0$. Since η can be arbitrarily small, we have $\mathrm{NUC}(\varepsilon) \geq \beta$.

(3) This is a direct result of (2) and Theorem 2.11. □

The following is the similar relationship between UKK spaces and NUC spaces:

Corollary 2.13 ([3]). *Let X be a Banach space. Then X is an* NUC *space if and only if X is a* UKK *space and reflexive.*

Proof. This is a direct result of Theorem 2.12. □

The following result is closely related to the one in [17].

Theorem 2.14. *Let X be a Banach space. If* $\mathrm{UKK}(1^-) > 0$, *then X has weak normal structure.*

Proof. By the assumption, there is $\eta \in (0, 1)$ such that $\mathrm{UKK}(\frac{1-\eta}{1+\eta}) > \frac{\eta}{1+\eta}$. Suppose that X does not have weak normal structure. By Lemma 2.2, there is a weak null sequence $\{x_n\} \subseteq S(X)$ satisfying the condition:

$$1 - \eta < \|x_n - x_m\| < 1 + \eta \quad \text{for } n \neq m.$$

Consider the sequence

$$y_n \equiv \frac{x_1 - x_{n+1}}{1 + \eta}.$$

Then $\{y_n\} \subseteq B(X)$ and $\|y_m - y_n\| = \frac{\|x_{n+1} - x_{m+1}\|}{1+\eta} > \frac{1-\eta}{1+\eta}$ for $m \neq n$. So $\mathrm{sep}(y_n) \geq \frac{1-\eta}{1+\eta}$. It is easy to see that $y_n \xrightarrow{w} \frac{x_1}{1+\eta}$. This implies that $\mathrm{UKK}(\frac{1-\eta}{1+\eta}) \leq \frac{\eta}{1+\eta}$, a contradiction. □

Corollary 2.15 ([3]). *If X is an* NUC *space, then X has normal structure.*

Corollary 2.16 ([3]). *If X is a* UKK *space, then X has weak normal structure.*

In 2007, Saejung proved the following result:

Lemma 2.17 ([18]). *If X is a Banach space with $B(X^*)$ weak* sequentially compact and it fails to have weak normal structure, then for any $\varepsilon > 0$ there are $\{x_1, x_2, \ldots, x_n\} \subseteq S(X)$ and $\{f_1, f_2, \ldots, f_n\} \subseteq S(X^*)$ such that*

(a) $|\|x_i - x_j\| - 1| < \varepsilon$, *for all* $i \neq j$;
(b) $\langle x_i, f_i\rangle = 1$, *for all* $1 \leq i \leq n$; *and*
(c) $|\langle x_j, f_i\rangle| < \varepsilon$, *for all* $i \neq j$.

This result can be extended as follows:

Lemma 2.18. *If X is a Banach space with $B(X^*)$ weak* sequentially compact and it fails to have weak normal structure, then for any $\varepsilon > 0$ there are two sequences $\{x_n\} \subseteq S(X)$ and $\{f_n\} \subseteq S(X^*)$ such that*

(a) $|\|x_i - x_j\| - 1| < \varepsilon$, *for all* $i \neq j$;
(b) $\langle x_i, f_i \rangle = 1$, *for all* $i \in \mathbb{N}$;
(c) $|\langle x_j, f_i \rangle| < \varepsilon$, *for all* $i \neq j$; *and*
(d) $\|f_i - f_j\| > 1 - \varepsilon$, *for all* $i \neq j$.

Proof. Let $\varepsilon > 0$, from Lemma 2.2 and the assumptions, for $\eta = \frac{\varepsilon}{2}$ we can find sequences $\{x_n\} \subseteq S(X)$, $\{f_n\} \subseteq S(X^*)$ and $f \in B(X^*)$, such that

(a) $|\|x_i - x_j\| - 1| < \eta$, where $i \neq j$;
(b) $\langle x_i, f_i \rangle = \|x_i\| = 1$, where $1 \le i < \infty$;
(c) $x_n \xrightarrow{w} 0$ and
(d) $f_n \xrightarrow{w^*} f$.

Since $x_n \xrightarrow{w} 0$, we can find a subsequence $1 < k_1 < k_2 < \cdots < k_n < \cdots$ such that $|\langle x_m, f_1 \rangle| < \eta$ for $m > k_1$, $|\langle x_m, f_{k_1} \rangle| < \eta$ for $m > k_2$, and so on.

So, the subsequence $1 < k_1 < k_2 < \cdots < k_n < \cdots$ satisfies $|\langle x_{k_j}, f_{k_i} \rangle| < \eta$ for $i < j$.

Without loss of generality, we still use $\mathbb{N}$ to denote the subsequence $1 < k_1 < k_2 < \cdots < k_n < \cdots$. So we have $|\langle x_j, f_i \rangle| < \eta$ for $i < j$.

Since $f_n \xrightarrow{w^*} f$, we can find a subsequence $1 < k_1 < k_2 < \cdots < k_n < \cdots$ such that $|\langle x_1, f_m - f \rangle| < \eta$ for $m > k_1$, $|\langle x_{k_1}, f_m - f \rangle| < \eta$ for $m > k_2$, and so on.

So, the subsequence $1 < k_1 < k_2 < \cdots < k_n < \cdots$ satisfies $|\langle x_{k_i}, f_{k_j} - f \rangle| < \eta$ for $i < j$.

Without loss of generality, we still use $\mathbb{N}$ to denote the subsequence $1 < k_1 < k_2 < \cdots < k_n < \cdots$. So we have $|\langle x_i, f_j - f \rangle| < \eta$ for $i < j$.

It follows that $|\langle x_i, f_j \rangle| \le |\langle x_i, f_j - f \rangle| + |\langle x_i, f \rangle| < 2\eta = \varepsilon$ if i and j large enough and $i < j$.

By deleting some terms in the front if necessary, we have $|\langle x_j, f_i \rangle| < \varepsilon$ for $i \neq j$.

Finally, $\|f_i - f_j\| \ge |\langle x_j, f_i - f_j \rangle| = |1 - \langle x_j, f_i \rangle| \ge 1 - |\langle x_j, f_i \rangle| > 1 - \varepsilon$ for $i \neq j$. □

Theorem 2.19. *Let X be a Banach space such that $B(X^*)$ is weak* sequentially compact. If* $\mathrm{UKK}^*((\frac{1}{\mu(X^*)})^-) > 1 - \frac{1}{\mu(X^*)}$ *for X^*, then X has weak normal structure.*

Proof. Suppose that X fails to have weak normal structure. From Lemma 2.18, for any $\eta > 0$, let the sequence $\{x_1, x_2, x_3, \ldots, x_n, \ldots\} \subseteq S(X)$ with $x_n \xrightarrow{w} 0$ and the sequence $\{f_1, f_2, \ldots, f_n, \ldots\} \subseteq S(X^*)$ with $f_n \xrightarrow{w^*} f$ satisfy four conditions there.

It is easy to see that from the definition of $\mu(X^*)$, we can find a subsequence $1 < k_1 < k_2 < \cdots < k_n < \cdots$ such that $1 - \eta < \|f_{k_i} - f_{k_j}\| < \mu(X^*) + \eta$ for $i \neq j$.

Without loss of generality, we still use $\mathbb{N}$ to denote the subsequence $1 < k_1 < k_2 < \cdots < k_n < \cdots$. So we have $1 - \eta < \|f_i - f_j\| < \mu(X^*) + \eta$ and $|\langle x_j, f_i \rangle| < \eta$ for all $i \neq j$.

Consider the sequence $\{g_1 = f_1 - f_2, g_2 = f_1 - f_3, \ldots, g_n = f_1 - f_{n+1}, \ldots\} \subseteq B_{\mu(X^*)+\eta}(0)$ for X^*. Then $f_n - f \xrightarrow{w^*} 0$. Moreover, for each i, $|\langle x_i, f \rangle| = \lim_{n\to\infty} |\langle x_i, f_n \rangle| < \eta$. So, $\|f_n - f\| \ge |\langle x_n, f_n - f \rangle| > 1 - \eta$ for each n. Note that $1 - \eta \le \|g_i - g_j\| = \|f_{j+1} - f_{i+1}\| \le \mu(X^*) + \eta$ for $i \neq j$, and $g_n \xrightarrow{w^*} f_1 - f$.

Let $z_i = \frac{g_i}{\mu(X^*)}$ for all $i \in N$. Then

$$\frac{1}{\mu(X^*)} - \frac{\eta}{\mu(X^*)} \le \|z_i - z_j\| = \frac{\|g_i - g_j\|}{\mu(X^*)} \le 1 + \frac{\eta}{\mu(X^*)} \quad \text{and} \quad z_n \xrightarrow{w^*} \frac{f_1 - f}{\mu(X^*)}.$$

From the definition of modulus of UKK^* for X^*, we have $\mathrm{UKK}^*(\frac{1}{\mu(X^*)} - \frac{\eta}{\mu(X^*)}) < 1 - \frac{1-\eta}{\mu(X^*)}$. Since η can be arbitrarily small,

$$\mathrm{UKK}^*\left(\left(\frac{1}{\mu(X^*)}\right)^-\right) < 1 - \frac{1}{\mu(X^*)}.$$

This completes the proof. □

Let $\mu(X^*) = 1$ in Theorem 2.19. Then we obtain the following result.

Theorem 2.20. *Let X be a Banach space such that $B(X^*)$ is weak* sequentially compact. If* $\mathrm{UKK}^*(1^-) > 0$ *for X^* and $\mu(X^*) = 1$, then X has weak normal structure.*

We now consider the ultraproduct spaces. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the set of natural numbers, we use $X_{\mathcal{U}}$ to denote the ultraproduct. For more details, see [19].

If X is super-reflexive, then $\widetilde{X^*} = (\widetilde{X})^*$ and X has uniform normal structure if and only if $\widetilde{X}$ has normal structure [19]. Since X can be embedded into $\widetilde{X}$, it is easy to see that $\mathrm{NUC}(\varepsilon)$ of X is greater than or equal to $\mathrm{NUC}(\varepsilon)$ of $\widetilde{X}$. By using these facts, we can prove the following result about uniform normal structure.

Theorem 2.21. *Let X be a super-reflexive Banach space with $\mathrm{NUC}(1^-) > 0$ for $\widetilde{X}$. Then X has uniform normal structure.*

Proof. If X is a super-reflexive Banach space but fails to have uniform normal structure, then $\widetilde{X}$ fails to have normal structure. From Theorem 2.14, we have $\mathrm{NUC}(1^-) > 0$ for $\widetilde{X}$. □

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